The flow of a compressible fluid with weak entropy changes

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SUMMARY

Perturbations of a given flow are considered, and the equations which govern the one-dimensional, non-steady flow of an inviscid, ideal compressible gas are linearized in the neighbourhood of this known solution, assumed isentropic, by a formal perturbation expansion. The perturbed flow is not assumed isentropic. Explicit solutions are obtained for a basic flow which is uniform or a centred simple wave and for an arbitrary simple wave if the perturbed flow is isentropic.

The perturbation of a uniform shock and perturbations in a shock tube lead to functional equations of a particular type, and a discussion of their solution is given.

A similar analysis is used to discuss the flow in a tube of slowly varying cross-section.

1. INTRODUCTION

The one-dimensional, non-steady flow of an inviscid compressible gas which, furthermore, is assumed to be ideal and polytropic with constant specific heats is governed by a system of non-linear, first-order partial differential equations which is always hyperbolic. Given an isentropic flow, a solution of the governing equations, this paper will discuss 'perturbations' of this basic flow; i.e. this flow is disturbed by some means, and it is desired to find the resultant flow.

To linearize in the neighbourhood of this known solution, the dependent variables are formally expanded as functions of a small parameter δ, viz.

$$u(x,t) = u_0(x,t) + \delta u_1(x,t) + \delta^2 u_2(x,t) + \ldots,$$
$$c(x,t) = c_0(x,t) + \delta c_1(x,t) + \delta^2 c_2(x,t) + \ldots,$$
$$s(x,t) = s_0 + \delta s_1(x,t) + \delta^2 s_2(x,t) + \ldots,$$

where u, c and s are respectively the particle velocity, the local speed of sound and the specific entropy, and the zero subscript denotes the basic (known) flow. When these expansions are substituted into the governing equations and grouped with respect to increasing powers of δ, the terms free of δ are satisfied identically, so that the resulting equations may be divided through by δ and, taking the limit as $\delta \to 0$, the equations governing the first approximation to the behaviour of the disturbed flow are obtained.

These equations are linear with the same characteristic surfaces as the original system and form a system of non-homogeneous first-order partial differential equations. It is convenient to attack the problem by first considering the associated homogeneous system, which has the physical significance that, to the order of terms retained (i.e. to the order of δ), the perturbed flow is still isentropic. The problem is consequently reduced to the determination of a particular solution of the non-homogeneous equations. Since the equations for the perturbations are linear, superposition of flows is possible.

Explicit solutions are obtained for a basic flow which is uniform or a centred simple wave, and for an arbitrary simple wave if the perturbed flow is isentropic. The application of these results to the perturbation of a uniform shock and perturbations in a shock tube lead to functional equations of a particular type, and a discussion of their solution is given in Appendices I and II. The flow in a long tube whose cross-section is made to undergo a small variation with x is treated by a similar analysis in part III.

Many authors have discussed the application of perturbation techniques to gas-dynamical problems, but, in general, have linearized in the neighbourhood of a constant state, with the result that the characteristics are rectilinear and parallel. For many flows, these characteristics are inadequate, and a more exact treatment of the characteristics is necessary. In the present paper, the question of correcting characteristics is not met, for the exact characteristics of the initial flow are always employed. A summary of § 2, § 3 and § 5 has already been given by Germain & Gundersen (1955).

PART I. GENERAL THEORY

2. The basic equations and solution for an originally uniform flow

The equation of state is

$$P = \exp\left[\frac{s-s^*}{c_v}\right]\rho^{\gamma},$$

and the equations which characterize the one-dimensional non-steady flow are

$$c_t + uc_x + \tfrac{1}{2}(\gamma-1)cu_x = 0,$$
$$u_t + uu_x + 2cc_x/(\gamma-1) - c^2 s_x/c_v\gamma(\gamma-1) = 0,$$
$$s_t + us_x = 0,$$

where P is the pressure, ρ the density, s^* the specific entropy at some reference state, γ the ratio of the specific heats at constant pressure c_p and at constant volume c_v, and the subscripts denote partial derivatives.

Throughout, it is assumed that an isentropic solution is known, namely, $u_0(x,t)$, $c_0(x,t)$, $s_0 =$ constant. A formal linearization in the neighbourhood of this known solution gives the following equations which govern the first approximation to the behaviour of the disturbed flow:

$$c_{1t}+u_0 c_{1x}+\tfrac{1}{2}(\gamma-1)c_0 u_{1x}+u_1 c_{0x}+\tfrac{1}{2}(\gamma-1)u_{0x}c_1 = 0, \tag{2.1}$$

$$\tfrac{1}{2}(\gamma-1)u_{1t}+\tfrac{1}{2}(\gamma-1)u_0 u_{1x}+c_0 c_{1x}+\tfrac{1}{2}(\gamma-1)u_{0x}u_1+c_1 c_{0x} = c_0^2 s_{1x}/2c_v\gamma, \tag{2.2}$$

$$s_{1t}+u_0 s_{1x} = 0. \tag{2.3}$$

These equations are linear with the same characteristic surfaces as the original system.

According to (2.3), s_1 remains constant along the particle paths of the given flow, i.e. along $dx/dt = u_0$. Also, $\rho_0(dx-u_0\,dt)$ is the exact differential of a function ψ_0 which, when equated to a constant, defines the particle paths. Consequently, the solution of (2.3) is

$$s_1 = \omega(\psi_0), \tag{2.4}$$

with ω an arbitrary function. It is convenient to define a new function $H_0(x,t)$ by

$$c_0^2 s_{1x} = \rho_0 c_0^2\,\omega'(\psi_0) = \gamma(\gamma-1)c_v H_0. \tag{2.5}$$

Therefore, the two equations which serve to determine u_1, c_1 are

$$c_{1t}+u_0 c_{1x}+\tfrac{1}{2}(\gamma-1)c_0 u_{1x}+u_1 c_{0x}+\tfrac{1}{2}(\gamma-1)u_{0x}c_1 = 0, \tag{2.6}$$

$$u_{1t}+u_0 u_{1x}+2(\gamma-1)^{-1}c_0 c_{1x}+u_{0x}u_1+2(\gamma-1)^{-1}c_{0x}c_1 = H_0. \tag{2.7}$$

It is convenient to introduce the characteristic parameters of the basic flow

$$\tfrac{1}{2}u_0+c_0/(\gamma-1) = \alpha, \qquad -\tfrac{1}{2}u_0+c_0/(\gamma-1) = \beta,$$

and the functions

$$\tfrac{1}{2}u_1+c_1/(\gamma-1) = A, \qquad -\tfrac{1}{2}u_1+c_1/(\gamma-1) = B.$$

The combinations $(2.6)/(\gamma-1)\pm\tfrac{1}{2}(2.7)$ give

$$A_t+(u_0+c_0)A_x+\tfrac{1}{2}[A(\gamma+1)+(\gamma-3)B]\alpha_x = \tfrac{1}{2}H_0, \tag{2.8}$$

$$B_t+(u_0-c_0)B_x+\tfrac{1}{2}[A(3-\gamma)-(\gamma+1)B]\beta_x = -\tfrac{1}{2}H_0. \tag{2.9}$$

A perturbation in which the perturbed flow is still isentropic corresponds to a solution of the homogeneous ($H_0 = 0$) system associated with the linear system (2.8) and (2.9).

For the case of an initially uniform flow, u_0 and c_0 are constants, and therefore $\alpha_x = \beta_x = 0$. The homogeneous system then admits the following general solution, in terms of two arbitrary functions F and G of one argument

$$A = F[x-(u_0+c_0)t], \qquad B = G[x-(u_0-c_0)t]. \tag{2.10}$$

In the non-isentropic case, from (2.4) and (2.5), H_0 is a function of $x-u_0 t$, and

$$H_0 = 2\chi'(x-u_0 t), \qquad c_0 A = c_0 B = \chi(x-u_0 t), \tag{2.11}$$

where χ is an arbitrary function, is a particular solution for which $u_1 = 0$; i.e. the addition of an entropy perturbation affects $c_1(x,t)$ but not $u_1(x,t)$.

3. Perturbation of a simple wave: general theory

For the case of the simple wave, one of the characteristic parameters of the basic flow remains constant, e.g. $\beta = \beta_0$. B is then determined from (2.9) and A from (2.8). The solution for the perturbation of an arbitrary simple wave will be given but, first, the centred simple wave will be discussed in some detail.

The wave is assumed to be centred at the origin and characterized by $\beta = \beta_0$. From the definition of the characteristic parameters,

$$u_0 = \alpha - \beta_0, \qquad c_0 = \tfrac{1}{2}(\gamma-1)(\alpha+\beta_0);$$

therefore

$$u_0 + c_0 = \tfrac{1}{2}(\gamma+1)\alpha + \tfrac{1}{2}(\gamma-3)\beta_0 = x/t,$$

$$\alpha = \frac{2}{\gamma+1}\left[\frac{x}{t} - \frac{\gamma-3}{2}\beta_0\right].$$

Substituting these results into (2.8) and (2.9) specialized to the case of an isentropic perturbed flow, the following equations are obtained:

$$tA_t + xA_x + \left[A + \frac{\gamma-3}{\gamma+1}B\right] = 0, \tag{3.1}$$

$$tB_t + \left[\frac{3-\gamma}{\gamma+1}x - 4\frac{\gamma-1}{\gamma+1}\beta_0 t\right]B_x = 0. \tag{3.2}$$

B will be determined first by solving (3.2), and, after this result has been substituted into (3.1), A may easily be determined. The characteristics of (3.2) may be written as

$$\frac{dt}{t} = \frac{(\gamma+1)dx}{(3-\gamma)x - 4(\gamma-1)\beta_0 t} = \left(\frac{\gamma+1}{3-\gamma}\right)\frac{d(x+2\beta_0 t)}{x+2\beta_0 t}, \tag{3.3}$$

and $dB = 0$. It follows from (3.3) that

$$t = K_1(x+2\beta_0 t)^{(\gamma+1)/(3-\gamma)}$$

with K_1 a constant; and since

$$c_0 = \frac{\gamma-1}{\gamma+1}\left[\frac{x+2\beta_0 t}{t}\right],$$

the representation of the curvilinear characteristics of the simple wave is

$$t = K_2(c_0 t)^{(\gamma+1)/(3-\gamma)}$$

with K_2 a constant, or

$$\rho_0 c_0 t^2 = \text{const.} \tag{3.4}$$

This form is not classical, but will be seen to be a very convenient one.

If F is an arbitrary differentiable function, it is convenient to write the solution of (3.2) as

$$B = 2(\rho_0 c_0)^{1/2} t\, F'(\rho_0 c_0 t^2). \tag{3.5}$$

Consequently, the solution of (3.1) is

$$At = \frac{3-\gamma}{\gamma+1}\frac{1}{(\rho_0 c_0)^{1/2}}F(\rho_0 c_0 t^2) + G(x/t), \tag{3.6}$$

where G is an arbitrary function.

The particle paths are given by $\rho_0 c_0 t = \text{const.}$, or

$$y \equiv c_0^2\, t^{2(\gamma-1)/(\gamma+1)} = \text{const.}$$

For the non-isentropic case, it is convenient to write

$$\tfrac{1}{2}H_0 = (2c_0/t)\omega_1'(y).$$

The following particular integral (to be added to the previous solution) is easily obtained:

$$A = B = \frac{\gamma+1}{2(\gamma-1)}\frac{c_0}{y}\omega_1(y),$$

i.e.

$$u_1 = 0, \qquad c_1 = \tfrac{1}{4}(\gamma+1)(c_0/y)\omega_1(y),$$

with ω_1 an arbitrary function and, as in the perturbation of a constant state, the addition of the entropy perturbation affects $c_1(x,t)$ but not $u_1(x,t)$. This question will be taken up in detail after the arbitrary simple wave has been discussed.

For the arbitrary simple wave (isentropic perturbed flow), let $x_0(z)$, $t_0(z)$ be the parametric representation of a curvilinear characteristic of the given flow. The wave may be represented by

$$x = x_0(z) + [u_0(z)+c_0(z)]\tau, \qquad t = t_0(z)+\tau, \tag{3.7}$$

$$-\tfrac{1}{2}u_0(z) + c_0(z)/(\gamma-1) = \beta_0. \tag{3.8}$$

Equation (2.9) reduces to

$$B_t + (u_0 - c_0)B_x = 0, \tag{3.9}$$

i.e. B is constant along any curvilinear characteristic. The characteristics of (3.9) are $dB = 0$ and $dx = (u_0 - c_0)\,dt$, or, from (3.7),

$$\frac{dx_0}{dz} + \tau\frac{d}{dz}(u_0+c_0) + \frac{d\tau}{dz}2c_0 = (u_0-c_0)\frac{dt_0}{dz};$$

hence, as (x_0, t_0) is a curvilinear characteristic, the following differential equation is obtained for $\tau(z)$:

$$2c_0\frac{d\tau}{dz} + \tau\frac{d}{dz}(u_0+c_0) = 0,$$

or, utilizing (3.8),

$$\frac{1}{\tau}\frac{d\tau}{dz} = -\frac{1}{2}\frac{\gamma+1}{\gamma-1}\frac{1}{c_0}\frac{dc_0}{dz}.$$

This has the solution

$$\rho_0 c_0 \tau^2 = \text{const.}, \quad \text{or } \rho_0 c_0 (t-t_0)^2 = \text{const.}$$

This result for the representation of the curvilinear characteristics of the simple wave could have been written down immediately by analogy with (3.4). The solution for B is

$$B = 2(\rho_0 c_0)^{1/2}(t-t_0)F'[\rho_0 c_0 (t-t_0)^2] \tag{3.10}$$

in terms of an arbitrary, differentiable function F. The characteristics of (2.8) may be written as

$$dt = \frac{dx}{u_0+c_0} = \frac{2\,dA}{[(3-\gamma)B-(\gamma+1)A]\alpha_x}. \tag{3.11}$$

The integral of the first equation of (3.11) has the first family of characteristics (rectilinear characteristics) for level curves. Let this first integral be

$$z(x,t) = \text{const.} \tag{3.12}$$

This function is given implicitly by

$$x - x_0(z) = [u_0(z) + c_0(z)][t - t_0(z)].$$

From the definition of the characteristic parameters, it follows that

$$\alpha = 2(\gamma+1)^{-1}[u_0 + c_0 - \tfrac{1}{2}(\gamma-3)]\beta_0.$$

Substitution in (3.11) yields the following differential equation for A:

$$\left[t - t_0 - \frac{2c_0(dt_0/dz)}{(dc_0/dz)}\frac{\gamma-1}{\gamma+1}\right]\frac{dA}{dt} + A = \frac{3-\gamma}{\gamma+1}B.$$

To integrate this equation, B is substituted from (3.10) and the first integral (3.12) is employed, i.e. z may be considered as constant. This yields another first integral, and the solution for $A(t,z)$ is

$$\left[t - t_0 - \frac{2c_0(dt_0/dz)}{(dc_0/dz)}\frac{\gamma-1}{\gamma+1}\right]A = G(z) + \\ + \frac{3-\gamma}{\gamma+1}\frac{1}{(\rho_0 c_0)^{1/2}}F[\rho_0 c_0(t-t_0)^2] \tag{3.13}$$

in terms of the arbitrary functions F and G. In the non-isentropic case, the problem is reduced to two ordinary differential equations, but it does not seem possible to solve these two equations explicitly in the general case. The question of actually determining these arbitrary functions in a specific problem will be discussed later.

4. Criterion for the existence of a particular solution $u_1 = 0$ in the non-isentropic case

In the non-isentropic perturbation of a constant state and of a centred simple wave, the addition of the entropy perturbation just affects the sound speed, whereas the particle velocity is the same for isentropic and non-isentropic perturbed flows, i.e. there exists a particular solution $u_1 = 0$. Unfortunately, such a nice result does not obtain for the arbitrary simple wave. It is of interest to determine under what conditions such a result is to be expected, for the answer is not at all obvious.

It is equivalent to look for a solution of (2.6) and (2.7) with $u_1 = 0$, viz.

$$c_{1t} + u_0 c_{1x} + \tfrac{1}{2}(\gamma-1)u_{0x}c_1 = 0, \tag{4.1}$$

$$(c_1 c_0)_x = (\rho_0 c_0^2/2c_v\gamma)\omega'(\psi_0), \tag{4.2}$$

$$d\psi_0 = \rho_0(dx - u_0\,dt).$$

Thus, it is clear that the problem consists of expressing the compatibility of equations (4.1) and (4.2), and the corresponding identities will determine which functions $u_0(x,t)$, $c_0(x,t)$ allow this compatibility.

The characteristics of (4.1) are

$$dt = \frac{dx}{u_0} = -\frac{2dc_1}{(\gamma-1)u_{0x}c_1}.$$

From the first two ratios, a first integral is

$$dx - u_0\, dt = 0, \qquad \text{or } \psi_0 = \text{const.}$$

A second one will be found easily since $c_0(x, t)$ is a particular solution of (4.1). This first integral is $c_1/c_0 = \text{const.}$ Hence, the solution of (4.1) is

$$c_0 c_1 = c_0^2 f(\psi_0) \tag{4.3}$$

where f is an arbitrary function. Substitution of (4.3) into (4.2) gives

$$\frac{c_{0x}}{\rho_0 c_0} = \frac{1}{2f}\left\{\frac{\omega'(\psi_0)}{2c_v \gamma} - f'\right\}.$$

Hence, the necessary and sufficient conditions for the existence of a particular solution $u_1 = 0$ is

$$\frac{c_{0x}}{\rho_0 c_0} = c^{-(\gamma+1)/(\gamma-1)} c_{0x} = \text{a function of } \psi_0.$$

By taking the material derivative, this condition becomes

$$u_{0xx} = 0.$$

Consequently, in order to have a particular solution with $u_1 = 0$, *it is necessary and sufficient that* $u_0(x, t)$ *is a linear function of* x. *Then* $c_1 = c_0 f(\psi_0)$ *with a convenient function* f. It is necessary to investigate all solutions of this form, i.e. $u_0 = a(t)x + b(t)$. This has been carried out but will not be included here.

5. An approximation for flows which differ only slightly from a uniform state

In general, it does not seem possible to solve (2.8) and (2.9) explicitly for an arbitrary initial flow. However, the preceding techniques may be utilized if the given flow does not differ greatly from a uniform state or a simple wave.

For example, consider the problem of a flow, which is nearly a uniform flow (u_0, c_0). Denote by u_1, c_1, s_1 and u_2, c_2, s_2 the terms of first and second order obtained by a formal perturbation method. Let

$$\bar{u}_0 = u_0 + u_1, \quad \bar{c} = c_0 + c_1 + c_2, \quad \bar{c}_0 = c_0 + c_1, \quad \bar{u} = u_0 + u_1 + u_2, \quad \bar{s} = s_1 + s_2.$$

By termwise addition of the equations determining u_1, c_1, s_1 and u_2, c_2, s_2 and neglecting terms of third order (e.g. $u_1 c_2$, $u_2 c_1$), the following equations are obtained:

$$\left.\begin{aligned} \bar{c}_t + \bar{u}_0 \bar{c}_x + \tfrac{1}{2}(\gamma - 1)\bar{c}_0 \bar{u}_x &= 0, \\ \bar{u}_t + \bar{u}_0 \bar{u}_x + 2(\gamma - 1)^{-1} \bar{c}_0 \bar{c}_x &= \bar{c}_0^2 \bar{s}_x / \gamma(\gamma - 1) c_v, \\ \bar{s}_t + \bar{u}_0 \bar{s}_x &= 0. \end{aligned}\right\} \tag{5.1}$$

Formally, this system of equations is identical to (2.1), (2.2) and (2.3) if the terms u_{0x} and c_{0x} are neglected. Consequently, the following principle has been established. *If* $\bar{u}_0$ *and* $\bar{c}_0$ *define the flow to the second order*, $\bar{u}$ *and* $\bar{c}$ *define the flow to the third order.*

The solution of the system (5.1) is immediate. With the introduction of the quantities A, B, α and β, the system may be written as

$$A_\beta = \tfrac{1}{2} H_0 t_\beta, \qquad B_\alpha = -\tfrac{1}{2} H_0 t_\alpha,$$

and may be solved by quadratures. In the isentropic case, $A = F(\alpha)$, $B = G(\beta)$ where F and G are two arbitrary functions. Thus, it is clear that the analysis is exactly that which is met in the classical theory (the Euler–Poisson equation).

Note that the neglect of the terms u_{0x} and c_{0x} does not affect the principal part of each differential equation, so that the characteristics remain unchanged.

6. Isentropic perturbation of an arbitrary simple wave: determination of arbitrary functions

In § 3, the perturbation of a simple wave was solved in terms of two arbitrary functions. The present and subsequent section are devoted to the question of the determination of these arbitrary functions. It will be shown that, at least theoretically, no difficulties are met.

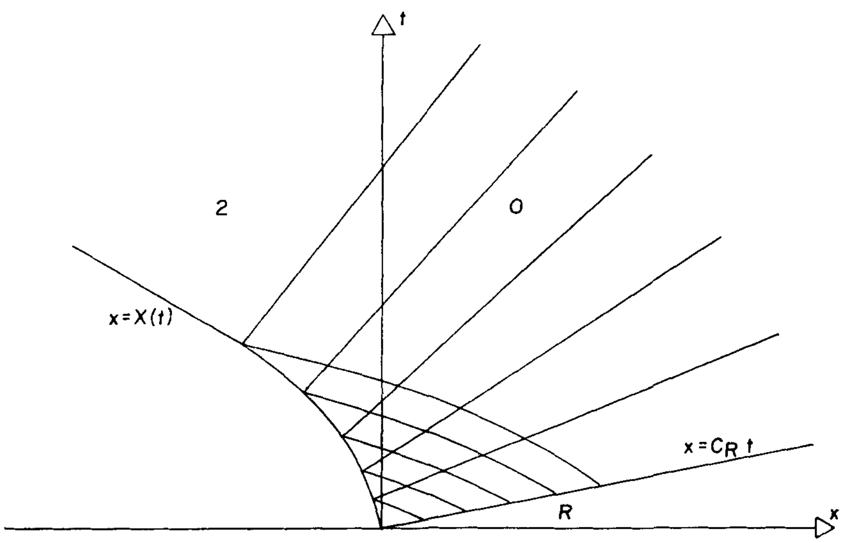

Figure 1.

Consider a tube of gas, initially at rest with constant sound speed c_R and constant density ρ_R, from which a piston, originally at rest, is withdrawn with increasing speed until a constant final velocity is attained. A simple wave, which advances with the speed c_R into the gas at rest, is generated. The (x, t)-representation of the flow is shown in figure 1. The region $x > c_R t$, denoted in figure 1 by the symbol R, is a constant state with $(u, c) = (0, c_R)$. The flow due to the receding motion of the piston is restricted to the region $x \leqslant c_R t$, and the region denoted by O is a simple wave. As the piston attains its final velocity, the flow becomes a constant state, denoted by the symbol 2, with particle velocity equal to the piston velocity. This is a classical problem, and the solution is well known.

It is now assumed that a small perturbation takes place and that the resultant flow is still isentropic. This problem was solved in § 3 in terms of two arbitrary functions. The discussion of this section will be in rather general terms without concern for computational details, for it is the basic principles, and not the details, which are important here; they will provide the basis for the solution of the perturbation of the centred simple wave.

Let the following initial conditions be given for $t = 0$:

$$u = \delta g(x), \qquad c = c_R + \delta h(x),$$

where g and h are known functions.

From the perturbation of a constant state (§ 2), the solution for $x \geqslant c_R t$ is

$$\bar{u}_R = \eta(x - c_R t) - \xi(x + c_R t), \qquad 2(\gamma - 1)^{-1}\bar{c}_R = \eta(x - c_R t) + \xi(x + c_R t),$$

where

$$\eta(x) = \tfrac{1}{2}g(x) + h(x)/(\gamma - 1), \qquad \xi(x) = -\tfrac{1}{2}g(x) + h(x)/(\gamma - 1),$$

and perturbations are denoted by a bar. Also, let the piston velocity be

$$u = \dot{x} = \dot{X}(t) + \delta\dot{\bar{X}}(t).$$

The rectilinear characteristics of the simple wave are given by $z =$ const., and the curvilinear cross-characteristics by

$$\rho_0 c_0 (t - t_0)^2 = a,$$

where a is a constant. Recall that $[x_0(z), t_0(z)]$ is the parametric representation of a non-linear characteristic of the given simple wave.

The function z was defined implicitly, and it is assumed that this relation has been inverted to give $z(x, t)$. This value is substituted into $\rho_0(z)$, etc., which gives functions which will be denoted by $\rho_{01}(x, t)$, etc. These are substituted into (3.10) and (3.13), and the resultant functions are called $B = B_1(x, t)$, $A = A_1(x, t)$, $G(z) = G_1(x, t)$ and F_1.

The perturbations introduced at $t = 0$ travel along the characteristics in region R to the first characteristic of the simple wave, viz. $x = c_R t$. At a point $(x_1, t_1) = (c_R t_1, t_1)$ of this characteristic, the perturbations are

$$A_1(x_1, t_1) = \eta(0), \qquad B_1(x_1, t_1) = \xi(2c_R t_1). \tag{6.1}$$

This perturbation $(\bar{u}_R^1, \bar{c}_R^1)$ travels along the non-linear characteristic through the point (x_1, t_1) to the piston curve. Let the equation of this non-linear characteristic be

$$\rho_{01} c_{01} (t - t_{01})^2 = a_1, \tag{6.2}$$

and let the intersection of (6.2) with the piston path, $x = X(t)$, be (x_1^1, t_1^1). The conditions (6.1) on $x = c_R t$ allow the determination of F_1 from

$$2(\rho_{01} c_{01})^{1/2}(t - t_{01}) F_1'[\rho_{01} c_{01}(t - t_{01})^2] = \xi(2c_R t_1). \tag{6.3}$$

Thus F_1, and therefore B_1, is known throughout the simple wave region.

In general, $\bar{u}_0 = A_1 - B_1$; therefore, at the point (x_1^1, t_1^1) of the piston curve, we have

$$\dot{\bar{X}}_1(t_1^1) = A_1(x_1^1, t_1^1) - B_1(x_1^1, t_1^1). \tag{6.4}$$

Now let

$$D_1^1 = \left[t - t_0 - 2c_0 \frac{\gamma-1}{\gamma+1} \frac{(dt_0/dz)}{(dc_0/dz)}\right]_{(x_1^1, t_1^1)}$$

As the first and last terms in (6.4) are known, this gives a condition for the determination of $G_1(x_1^1, t_1^1)$:

$$G_1(x_1^1, t_1^1) = D_1^1[\dot{X}_1(t_1^1) + B_1(x_1^1, t_1^1)] + \\ + \frac{\gamma-3}{\gamma+1} \frac{1}{(\rho_{01}^1 c_{01}^1)^{1/2}} F_1[\rho_{01}^1 c_{01}^1 (t - t_{01}^1)^2]. \quad (6.5)$$

Clearly, this analysis applies at each point of the first characteristic of the simple wave, i.e. at each point the perturbation travels along the non-linear characteristic through the point to the piston curve. Consequently, (6.5) holds, with appropriate changes in the arguments, all along the portion of the piston curve in the simple wave region and allows the determination of $G_1(x, t)$. From classical theory, this is a well-set problem (in Picard's terminology, a third problem) as the value of B_1 is known along the characteristic $x = c_R t$ and the value of A_1 is known along the piston curve which has time-like orientation.

The function G_1 determines the perturbations which travel along the rectilinear characteristics of the simple wave. The perturbed flow in region 2 is determined by data along the last characteristic of the simple wave and on the rectilinear portion of the piston curve.

7. Isentropic perturbation of a centred simple wave

Consider the same original situation as in § 6, but suppose now that the acceleration of the piston from rest to a constant terminal velocity $u_1 = -V$ ($V > 0$) takes place instantaneously. The rectilinear characteristics of the rarefaction wave, generated by this piston motion, all pass through the origin as depicted in figure 2.

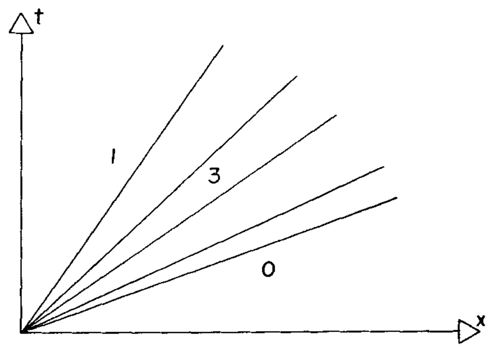

Figure 2.

On this known initial flow, the following initial conditions for the perturbed flow are superposed: $\bar{u}_0 = 0$, $t = 0$; $\bar{c}_0 = (\gamma-1)f(x)$, $x > 0$; $\bar{u}_0(0,0) = 0$; $\bar{c}_0(0,0) = (\gamma-1)f(0)$. The analysis of § 6 will be used to determine the solution of the present problem. The subsequent analysis will also give information about the value of $\bar{c}_0(0,0)$.

The utilization of the solution for the perturbation of a constant state and initial conditions gives

$$A = f(x - c_R t), \qquad B = f(x + c_R t). \tag{7.1}$$

The application of (7.1) on $x = c_R t$ and substitution in (3.5) and (3.6) gives

$$G(c_R) = f(0)t + \frac{\gamma-3}{\gamma+1}\frac{1}{(\rho_R c_R)^{1/2}} F(\rho_R c_R t^2),$$
$$f(2c_R t) = 2(\rho_R c_R)^{1/2} t\, F'(\rho_R c_R t^2).$$

In the limit as $t \to 0$,

$$\left.\begin{aligned} G(c_R) &= \frac{\gamma-3}{\gamma+1}\frac{1}{(\rho_R c_R)^{1/2}} F(0), \\ f(0) &= 0, \end{aligned}\right\} \tag{7.2}$$

where it is assumed that f is continuous and that F' is bounded near $t = 0$. This last statement implies that the perturbations are bounded near $t = 0$.

Equation (7.2) says that, for the present problem, it is necessary that $\bar{c}_0(0,0) = 0$, i.e. both perturbation velocities must vanish at the origin where the basic flow is discontinuous.

The present problem is clearly a limiting case of the problem of § 6. The G-function, a function of x/t, determines the perturbations which travel along the rectilinear characteristics of the simple wave. It is determined by the perturbations carried to the piston curve along the curvilinear cross-characteristics of the simple wave. In the limit as the rectilinear characteristics form a pencil of lines through the origin, the G-function will be determined by the perturbations at the origin only. As the perturbations are zero at the origin, it is clear that G is at most a constant, and this constant will be chosen as zero for convenience.

It is easily found that

$$F(\rho_3 c_3 t^2) = (\rho_R c_R)^{1/2} \int_0^{(\rho_3 c_3/\rho_R c_R)^{1/2} t} f(2c_R y)\, dy.$$

The solution in the simple wave is now known, and the perturbed flow in region 1 is determined by conditions along the last characteristic of the simple wave and along the piston.

PART II. APPLICATIONS

As a first application, the perturbation of a shock produced by a uniform compressive motion of a piston will be considered. A piston, initially at rest, is at $t = 0$ suddenly pushed with constant velocity into a gas which is also initially at rest. A uniform shock is produced, which moves with constant velocity into the gas, and the gas behind the shock is in a steady state with gas velocity equal to the piston velocity. The state of rest in front of the shock is denoted by R, and the constant state behind the shock is denoted by 0. The shock velocity is given by

$$W_0 = \tfrac{1}{4}(\gamma+1)u_0 + [c_R^2 + \{(\gamma+1)u_0/4\}^2]^{1/2}.$$

Suppose that the piston is given a small prescribed perturbation for $t \geqslant t_1$. This will perturb the shock locus, and, as this is the perturbation of a strong shock, the flow behind the shock will no longer be isentropic; thus the entropy perturbations must be considered. It is convenient to solve this problem in two steps. First, the inverse problem of the determination of the perturbed piston path when the shock is given a small perturbation is solved, and, second, this solution is used to solve the direct problem.

8. The inverse problem

Suppose that, from some particular time t_2, the shock is given a small prescribed perturbation; i.e. let the shock velocity be $W = W_0$ for $t < t_2$ and $W = W_0 + \epsilon(t)$ for $t \geqslant t_2$, where $\epsilon(t)$ is a known function sufficiently small so that terms of higher order than the first may be neglected. This perturbation will cause a corresponding perturbation of the particle velocity; let the new particle velocity near the shock be $u = u_0 + \eta(t)$, where $\eta(t)$ is also small and must be a function of $\epsilon(t)$ as will be shown.

The perturbations of u_0, c_0 and s_0, which will be denoted by $\bar{u}_0$, $\bar{c}_0$ and $\bar{s}_0$, will be found in terms of $\epsilon(t)$ along the shock. With these boundary conditions, the solution for the non-isentropic perturbation of a constant state may be utilized to find the perturbed flow behind the shock. Then, $\bar{u}_0(u_0 t, t)$ gives the perturbation of the piston velocity.

It is convenient to tabulate many of the standard relations between the flow parameters in perturbation form. Appendix III contains some of the more important relations.

By the use of the Prandtl relation in perturbation form (Appendix III, relation VIII), it follows that

$$\bar{u}_0 = 2K_1\epsilon(t), \qquad \bar{c}_0 = (\gamma-1)K_2\epsilon(t),$$

where

$$K_1 = \tfrac{1}{2}[2(1-\theta-u_0/W_0], \qquad K_2 = 2(\gamma+1)^{-1}(W_0-u_0)/c_0 + u_0^2/2W_0c_0,$$
$$\theta = (\gamma-1)/(\gamma+1).$$

Consequently,

$$A = (K_1+K_2)\epsilon(t) \equiv K_3\epsilon(t), \qquad B = (K_2-K_1)\epsilon(t) \equiv K_4\epsilon(t) \qquad (8.1)$$

on the shock.

From relation V of Appendix III,

$$\frac{\bar{\rho}_0}{\rho_0} = \left[\frac{u_0-(1-\theta)W_0}{W_0(u_0-W_0)}\right]\epsilon(t);$$

and from relation I of Appendix III,

$$\bar{s}_0 = \left\{\frac{2c_v}{c_0}(\gamma-1)K_2 - \frac{(\gamma-1)}{W_0}c_v\frac{[u_0-(1-\theta)W_0]}{(u_0-W_0)}\right\}\epsilon(t) \equiv K_5\epsilon(t). \qquad (8.2)$$

In general, from (2.11) and (2.5),

$$\bar{s}_0 = 2c_v\gamma(\gamma-1)c_0^{-2}\chi(x-u_0 t)$$

which implies, from (8.2), that

$$\chi(x-u_0 t) = \frac{K_5 c_0^2}{2c_v\gamma(\gamma-1)}\epsilon\left[\frac{x-u_0 t}{W_0-u_0}\right] \tag{8.3}$$

throughout region O.

By the use of the general solution for the perturbation of a constant state and the conditions (8.1) and (8.3), and applying the boundary conditions on the unperturbed shock locus*, the following solution is obtained:

$$A = \frac{c_0 K_5}{2c_v\gamma(\gamma-1)}\epsilon\left[\frac{x-u_0 t}{W_0-u_0}\right]+\left[K_3-\frac{c_0 K_5}{2c_v\gamma(\gamma-1)}\right]\epsilon\left[\frac{x-(u_0+c_0)t}{W_0-u_0-c_0}\right],$$

$$B = \frac{c_0 K_5}{2c_v\gamma(\gamma-1)}\epsilon\left[\frac{x-u_0 t}{W_0-u_0}\right]+\left[K_4-\frac{c_0 K_5}{2c_v\gamma(\gamma-1)}\right]\epsilon\left[\frac{x-(u_0-c_0)t}{W_0-u_0+c_0}\right].$$

Hence

$$\bar{u}_0 = \left[K_3-\frac{c_0 K_5}{2c_v\gamma(\gamma-1)}\right]\epsilon\left[\frac{x-(u_0+c_0)t}{W_0-u_0-c_0}\right]+ \\ +\left[\frac{c_0 K_5}{2c_v\gamma(\gamma-1)}-K_4\right]\epsilon\left[\frac{x-(u_0-c_0)t}{W_0-u_0+c_0}\right] \tag{8.4}$$

is the particle velocity perturbation throughout region O, and, in particular its value on the piston path is obtained by setting $x = u_0 t$ in (8.4). The perturbed piston and shock paths are obtained by simple integration.

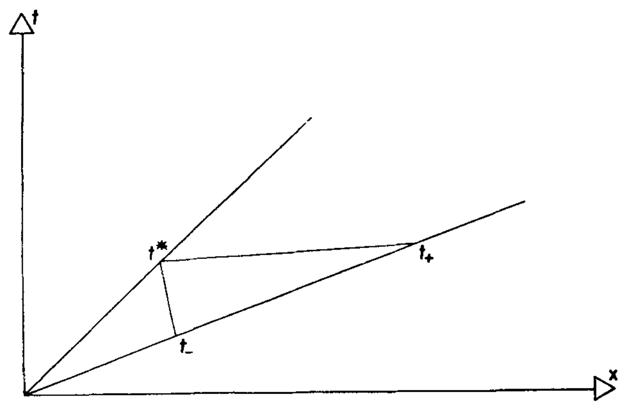

Figure 3.

It is not immediately obvious how the characteristics should be oriented. However, in figure 3, consider an arbitrary point on the piston curve

* The error thus introduced is of the order of the neglected terms and therefore neglible, viz. if the original shock locus is $x = W_0 t$, the new shock locus is $x = W_0 t+\int_{t_s}^{t} \epsilon(y)\,dy$, and

$$\eta\left[W_0 t+\int_{t_s}^{t}\epsilon(y)\,dy\right] = \eta(W_0 t)+\eta'(W_0 t)\int_{t_s}^{t}\epsilon(y)\,dy+\dots.$$

(e.g. t^*), and let t_- be the intersection of the shock locus and the characteristic with slope $u_0 - c_0$ through this point. The characteristic with slope $u_0 + c_0$ through t^* intersects the shock locus in a point which will be denoted by t_+. These points are given by

$$t_+ = c_0 t^*/(u_0 + c_0 - W_0), \qquad t_- = c_0 t^*/(c_0 - u_0 + W_0).$$

Consequently, from (8.4) with $x = u_0 t$, it is seen that a perturbation at a particular point of the piston curve is determined by the perturbation at two points of the shock curve with the depicted orientation. Hence, the (x, t)-representation is as in figure 4. The region in which the flow is perturbed is cross-hatched. The relation between t_1 and t_2 is

$$t_2 = c_0 t_1/(u_0 + c_0 - W_0).$$

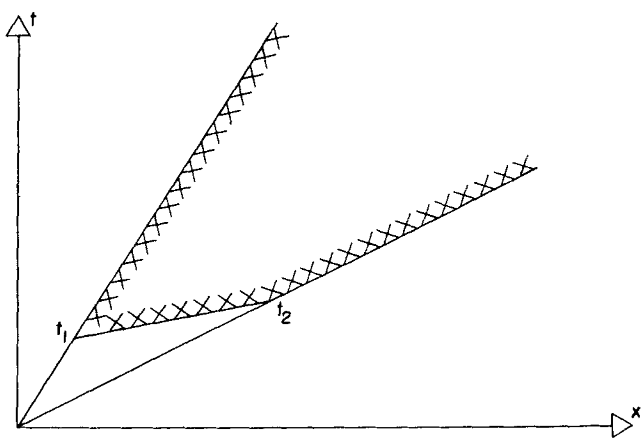

Figure 4.

9. Direct problem

From the solution of the inverse problem, the solution of the direct problem is easily obtained, and the solution is given by a uniformly convergent infinite series. Let $\eta(t)$ be the prescribed perturbation of the piston velocity, and $\epsilon(t)$ the perturbation of the shock velocity. From (8.4), we get

$$\eta(t) = \left[K_3 - \frac{c_0 K_5}{2c_v \gamma(\gamma-1)}\right]\epsilon\left[\frac{c_0 t}{c_0 + u_0 - W_0}\right] + \left[\frac{c_0 K_5}{2c_v \gamma(\gamma-1)} - K_4\right]\epsilon\left[\frac{c_0 t}{W_0 - u_0 + c_0}\right]$$
$$\equiv R_1 \epsilon\left[\frac{c_0 t}{c_0 - u_0 + W_0}\right] + R_2 \epsilon\left[\frac{c_0 t}{u_0 + c_0 - W_0}\right]. \quad (9.1)$$

Figure 5 illustrates the scheme to be utilized. The lines drawn between the piston and shock paths are the characteristics, the odd-numbered lines having slope $(u_0 + c_0)$ and the even-numbered lines having slope $(u_0 - c_0)$.

Now, η evaluated at the point 1 is given in terms of multiples of ϵ evaluated at the points 1′ and 2′. $\eta(2)$ is given in terms of multiples of $\epsilon(2')$ and $\epsilon(3')$, etc. $\epsilon(1')$ will be determined. Symbolically, take the first-mentioned relation (1, 1′, 2′), equation (9.2), and for 2′ substitute its equivalent from equation (9.3). This gives a relation (1, 3, 1′, 4′). By continuing this process, a relation of the form (1, 3, 5, 7, ..., n, 1′, $[n+1]'$) for n odd is obtained. By indefinite continuation of this process, a series representation is obtained for ϵ in terms of multiples of η evaluated at the points K, where $K = 1, 3, 5, \ldots$. The general term of this series is determined and proved by induction, and the convergence is established.

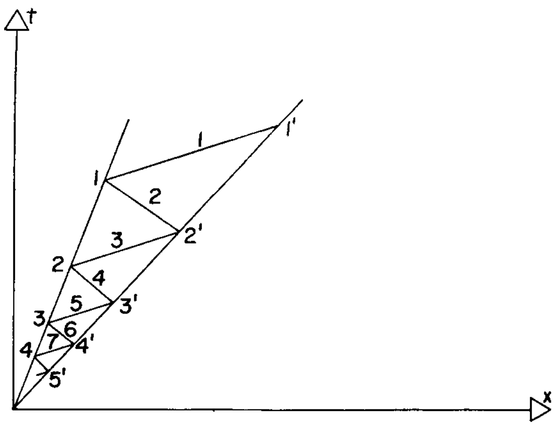

Figure 5.

The coordinates of the point 1′ are chosen to be $(x, t) = (W_0 t^*, t^*)$, and the following notation will be employed:

$$u_0 + c_0 - W_0 \equiv r, \qquad W - u_0 + c_0 \equiv q, \qquad c_0 t^* \equiv t_1.$$

By induction,

$$\eta(1) = R_1 \epsilon(r t_1/q^2) + R_2 \epsilon(t_1/r) \tag{9.2}$$

$$\eta(2) = R_1 \epsilon(r^2 t_1/q^3) + R_2 \epsilon(t_1/q)$$

$$\eta(3) = R_1 \epsilon(r^3 t_1/q^4) + R_2 \epsilon(r t_1/q^2) \tag{9.3}$$

$$\cdots\cdots\cdots$$

$$\eta(n) = R_1 \epsilon(r^n t_1/q^{n+1}) + R_2 \epsilon(r^{n-2} t_1/q^{n-1}). \tag{9.4}$$

From (9.2),

$$\epsilon(t_1/r) = \eta(1)/R_2 - (R_1/R_2)\epsilon(r t_1/q^2).$$

The last term is replaced by its equivalent from (9.3), with the result

$$\epsilon(t_1/r) = \eta(1)/R_2 - (R_1/R_2^2)\eta(3) + (R_1/R_2)^2 \epsilon(r^3 t_1/q^4).$$

For the first n terms, the series is

$$\epsilon(t_1/r) = \sum_{K=1}^{n} (-1)^{K+1} R_1^{K-1} R_2^{-K} \eta(2K-1) + (-1)^n (R_1/R_2)^n \epsilon(r^{2n-1} t_1/q^{2n}). \tag{9.5}$$

Mathematical induction is used to prove this assertion, which is certainly true for $K = 1$. Assume it is true for $K = n$. From (9.4), it follows that

$$\epsilon(r^{2n-1} t_1/q^{2n}) = \eta(2n+1)/R_2 - (R_1/R_2)\epsilon(r^{2n+1} t_1/q^{2n+2}). \tag{9.6}$$

Substitution of (9.6) into (9.5) gives

$$\epsilon(t_1/r) = \sum_{K=1}^{n+1} \eta(2K-1)(-1)^{K+1}R_1^{K-1}R_2^{-K} + (-1)^{n+1}(R_1/R_2)^{n+1}\epsilon(r^{2n+1}t_1/q^{2n+2}). \tag{9.7}$$

Therefore, the assumption that (9.5) is true for n implies it is true for $n+1$. Hence, (9.5) is true for all n.

Since it is assumed that all perturbations are small, the assumption is made that ϵ and η are uniformly bounded. With this reasonable assumption and the easily-seen fact that $R_1/R_2 < 1$, it is readily established that, as $n \to \infty$, the series is uniformly convergent. Hence

$$\epsilon(t_1/r) = \sum_{K=1}^{\infty} (-1)^{K+1}R_1^{K-1}R_2^{-K}\eta(2K-1).$$

Functional equations of the type (9.1) are treated from a general point of view in Appendix I.

10. Perturbations in a shock tube

The orthodox shock tube consists of a tube divided by a membrane (say at $x = 0$) which separates two fluids, not necessarily the same, originally at rest but not at the same pressure. This is one frequently used device for generating constant shocks in a long tube. It will be assumed that the fluid in the region $x > 0$ is at the higher pressure. If, at $t = 0$, the membrane is removed instantaneously, the fluid in the region $x > 0$ will expand, while the fluid in the region $x < 0$ will be compressed. Under these assumptions, the expansion takes place through a centred rarefaction wave, and the compression across a uniform shock. In general, there will be a contact discontinuity which separates the two portions of fluid as depicted in figure 6.

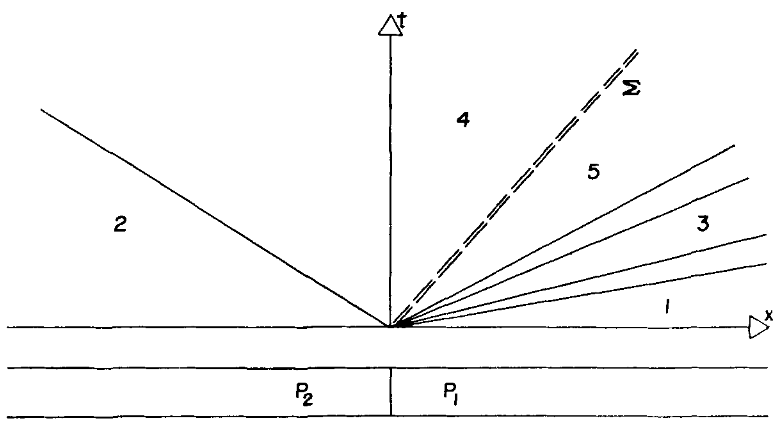

Figure 6.

On this basic flow, which is assumed to be completely known, a perturbation is imposed. The perturbation conditions are not assumed to be the same for the two portions of the fluid. In particular, the isentropic character of the initial flow will not be retained for the region $x > 0$. Let these conditions be: for $t = 0$ and $x > 0$, $u = \delta g(x)$, $c = c_1 + \delta h(x)$, $s = s_0 + \delta f(x)$; for $t = 0$ and $x < 0$, $u = \delta n(x)$, $c = c_2 + \delta m(x)$, $s = s_0$. Here c_1, c_2 and s_0 are constants, and g, h, f, m and n are known functions. For continuity at $x = 0$, it is assumed that $g(0) = n(0)$, $h(0) = m(0)$ and $f(0) = 0$. The contact discontinuity is given by $x = Kt$.

It is not possible to solve independently for the perturbed flows on the two sides of the contact discontinuity. For the region $x > 0$, no difficulty is encountered in regions 1 and 3. The solution in 3 gives conditions on the last characteristic of the simple wave, but these are insufficient to determine the flow in region 5. However, if on the contact discontinuity $\bar{u}_5$ is specified as $\xi_1(t)$, which is not known, the flow in region 5 can be easily determined in terms of $\xi_1(t)$. For the region $x < 0$, the flow will be non-isentropic since the shock is perturbed. By expressing $\bar{u}_4$, $\bar{c}_4$ and $\bar{s}_4$ in terms of the shock velocity perturbation $\bar{W}$, the flow in region 4 can be determined in terms of $\bar{W}$, which is not known.

Across the contact discontinuity, there is continuity of pressure and particle velocity, and it is easy to see that this implies continuity of the corresponding perturbations. This continuity condition yields two equations in $\xi_1(t)$ and $\bar{W}(t)$. By elimination of $\xi_1(t)$, a functional equation, treated in detail in Appendix II, is obtained for $\bar{W}$ and, once $\bar{W}$ has been determined, the solution of the problem is easily completed.

In terms of $\eta(x)$ and $\xi(x)$, as defined in § 6, the solution in region 1 is

$$\begin{aligned}\bar{u}_1 &= \eta(x-c_1t)-\xi(x+c_1t)-\tfrac{1}{2}\rho_1c_1f'(x-c_1t)+\tfrac{1}{2}\rho_1c_1f'(x+c_1t),\\ 2(\gamma-1)^{-1}\bar{c}_1 &= \eta(x-c_1t)+\xi(x+c_1t)-\tfrac{1}{2}\rho_1c_1f'(x-c_1t)-\\ &\qquad -\tfrac{1}{2}\rho_1c_1f'(x+c_1t)+\rho_1c_1f'(x),\\ \bar{s}_1 &= \gamma(\gamma-1)c_v\rho_1f(x).\end{aligned}$$

These perturbations travel along the characteristics in region 1 to the first characteristic of the centred simple wave, where they supply boundary conditions for determining the perturbed flow in region 3. On $x = c_1t$, the first characteristic of the simple wave is

$$\begin{aligned}B &= \xi(2c_1t)-\tfrac{1}{2}\rho_1c_1f'(2c_1t)+\tfrac{1}{2}\rho_1c_1f'(c_1t),\\ \bar{s}_{1x} &= \gamma(\gamma-1)c_v\rho_1f'(c_1t).\end{aligned}$$

Application of these conditions and the solution for the non-isentropic perturbation of a centred simple wave gives

$$\omega_1(c_3^2t^{2\theta}) = \theta c_1^3\rho_1\int_0^{(c_3/c_1)^{\theta}t/\theta} y^{2\theta}f'(c_1y)\,dy,$$

$$F(\rho_3c_3t^2)/(\rho_1c_1)^{1/2} = \int_0^{(\rho_3c_3/\rho_1c_1)^{1/2}t}\{\xi(2c_1z)-\tfrac{1}{2}\rho_1c_1f'(2c_1z)+\tfrac{1}{2}\rho_1c_1f'(c_1z)\}\,dz-$$
$$-\rho_1c_1^3\int_{z=0}^{t} z^{-2\theta}\int_{y=0}^{z} y^{2\theta}f'(c_1y)\,dydz.$$

Consequently, A, B and $\bar{s}_3$ are known throughout the simple wave. As the expressions are very complicated, let the values on the last characteristic of the simple wave be

$$A = \zeta_1(t), \qquad B = \zeta_2(t), \qquad \bar{s}_{3x} = \zeta_3(t),$$

where ζ_1, ζ_2 and ζ_3 are known functions. However, these conditions are not sufficient to determine the perturbed flow in region 5. This is clear, for it cannot be expected that data along one characteristic, but nothing on another curve, would yield a well-set problem. It is convenient to let $\bar{u}_5 = \xi_1(t)$ on $x = Kt$, the contact discontinuity, where, of course, $\xi_1(t)$ is not known at the present time. The perturbed flow in region 5 can now be easily found. Simple reasoning shows that $\bar{c}_5$ is a constant multiple of $\xi_1(t)$ plus and a known function and, therefore,

$$\bar{P}_5 = \nu\xi_1(t) + \Gamma(t)$$

where ν is a constant and Γ is a known function.

As the pressure and velocity are continuous across the contact discontinuity, the perturbations of these quantities are also continuous across this surface, so that

$$\bar{u}_4 = \xi_1(t), \qquad \bar{P}_4 = \nu\xi_1(t) + \Gamma(t)$$

on $x = Kt$.

For region 2, the solution is easily found. This gives $\bar{u}_2$ and $\bar{c}_2$ and, from Appendix III, all perturbed quantities in region 2 are easily determined.

From Appendix III, the values of the perturbed quantities in region 4 may be found just behind the shock in terms of $\bar{W}$. The analysis is simple though tedious, and simple reasoning shows that they have the following form:

$$\left.\begin{aligned} \bar{P}_4 = R_5 + R_6\bar{W}, \qquad \bar{\rho}_4 &= R_7 + R_8\bar{W}, \qquad \bar{c}_4 = R_9 + R_{10}\bar{W},\\ \bar{s}_4 = R_{11} + R_{12}\bar{W}, \qquad \bar{u}_4 &= R_{13} + R_{14}\bar{W},\\ A = R_{15} + R_{16}\bar{W}, \qquad B &= R_{17} + R_{18}\bar{W}, \end{aligned}\right\} \tag{10.1}$$

where all the R's are known, and odd-numbered subscripts denote functions of t while even-numbered subscripts denote constants.

Utilization of the solution for the non-isentropic perturbation of a constant state and the boundary conditions (10.1) gives

$$A = R_{19}\left[\frac{x-u_4t}{W-u_4}\right] + R_{21}\left[\frac{x-(u_4+c_4)t}{W-u_4-c_4}\right] + \\ + R_{22}\bar{W}\left[\frac{x-(u_4+c_4)t}{W-u_4-c_4}\right] + R_{20}\bar{W}\left[\frac{x-u_4t}{W-u_4}\right],$$

$$B = R_{19}\left[\frac{x-u_4t}{W-u_4}\right] + R_{23}\left[\frac{x-(u_4-c_4)t}{W-u_4+c_4}\right] + \\ + R_{24}\bar{W}\left[\frac{x-(u_4-c_4)t}{W-u_4+c_4}\right] + R_{20}\bar{W}\left[\frac{x-u_4t}{W-u_4}\right],$$

where all the R's are known.

Throughout the region 4, we have

$$\bar{p}_4 = \frac{2\gamma p_4}{c_4(\gamma-1)}\bar{c}_4 - \frac{p_4}{(\gamma-1)c_v}\bar{s}_4.$$

The continuity of $\bar{P}$ across the contact discontinuity gives an equation of the form

$$\xi_1(t) = R_{25}(t) + R_{22}\,\bar{W}\left[\left(\frac{K-u_4-c_4}{W-u_4-c_4}\right)t\right] + \\ + R_{24}\,\bar{W}\left[\left(\frac{K-u_4+c_4}{W-u_4+c_4}\right)t\right] + R_{26}\,\bar{W}\left[\left(\frac{K-u_4}{W-u_4}\right)t\right], \quad (10.2)$$

and the continuity of $\bar{u}$ across the contact discontinuity gives

$$\xi_1(t) = R_{27}(t) + R_{28}\,\bar{W}\left[\left(\frac{K-u_4-c_4}{W-u_4-c_4}\right)t\right] + R_{30}\,\bar{W}\left[\left(\frac{K-u_4+c_4}{W-u_4+c_4}\right)t\right]. \quad (10.3)$$

Division of (10.2) by (10.3) serves to eliminate $\xi_1(t)$, with the result

$$R_{32}\,\bar{W}\left[\left(\frac{K-u_4+c_4}{W-u_4+c_4}\right)t\right] + R_{34}\,\bar{W}\left[\left(\frac{K-u_4-c_4}{W-u_4-c_4}\right)t\right] + \\ + R_{36}\,\bar{W}\left[\left(\frac{K-u_4}{W-u_4}\right)t\right] = R_{31}(t),$$

where all the R's are known. Once $\bar{W}$ has been found, $\xi_1(t)$ is given immediately, and the solution is easily completed. This functional equation is discussed in Appendix II.

PART III. TUBE OF SLOWLY VARYING CROSS-SECTION

The flow in a long tube whose cross-section is made to undergo a small variation with x can be treated by a similar analysis. It is convenient to express the cross-section of the tube in the form $E(x) = E_0 + \delta E_1(x)$, where E_0 is the original uniform cross-sectional area and δ is a small parameter.

It will be assumed that the tube consists of two parts, one of uniform cross-section, and one of perturbed cross-section. A uniform shock, introduced in the tube where the cross-section is uniform, will travel along the tube and remain uniform until it reaches the transition section. The gas in front of the shock is assumed to be at rest, and the small effects due to the discontinuity in the slope of the tube at the transition section will be neglected. After the shock passes the transition section, it will be perturbed, and the flow behind the shock will necessarily be non-isentropic. The problem is to describe the flow behind the shock and to find the perturbed shock locus. The results are applied to several problems, and some comparisons are made with work by a previous author.

11. Derivation of equations which govern the flow

Clearly, the Euler and energy equations are not affected by the cross-sectional perturbation, but the continuity equation contains an additional term. The mass, contained in an arbitrary volume τ, may be written as

$$M = \int_\tau \rho \, d\tau = \int_x \rho E(x) \, dx.$$

The continuity equation, $DM/Dt = 0$, implies that

$$(\rho E)_t + (\rho E u)_x = 0.$$

Consequently, the basic equations are

$$u_t + u u_x + 2c_x/(\gamma - 1) = s_x c^2/c_v \gamma(\gamma - 1),$$
$$2Ec_t/(\gamma-1) + 2Euc_x/(\gamma-1) + cEu_x + cE_t + ucE_x = 0,$$
$$s_t + us_x = 0.$$

A formal linearization in the neighbourhood of an originally isentropic flow gives

$$u_0 u_{1x} + u_{1t} + 2c_0 c_{1x}/(\gamma - 1) + u_1 u_{0x} + 2c_{0x} c_1/(\gamma - 1) = c_0^2 s_{1x}/c_v \gamma(\gamma - 1), \quad (11.1)$$

$$2c_{1t}/(\gamma-1) + c_0 u_{1x} + 2u_0 c_{1x}/(\gamma-1) + c_1 u_{0x} + 2u_1 c_{0x}/(\gamma-1) + $$
$$+ u_0 c_0 E_{1x}/E_0 + 2c_{0t} E_1/E_0(\gamma-1) + c_0 u_{0x} E_1/E_0 +$$
$$+ 2u_0 c_{0x} E_1/E_0(\gamma-1) = 0, \quad (11.2)$$

$$s_{1t} + u_0 s_{1x} = 0. \quad (11.3)$$

It is thus clear that the cross-sectional perturbation merely introduces a non-homogeneous term into the basic perturbation equations which were previously derived, so that the mathematical results of the previous sections may be utilized, and the problem is reduced to the determination of a particular integral of (11.1), (11.2) and (11.3).

For the case of an initially uniform flow, (11.3) may be solved independently, and, utilizing the same procedure as employed in §2, the following equations are obtained:

$$A_t + (u_0 + c_0) A_x = -\tfrac{1}{2} u_0 c_0 E_{1x}/E_0 + \tfrac{1}{2} H_0,$$
$$B_t + (u_0 - c_0) B_x = -\tfrac{1}{2} u_0 c_0 E_{1x}/E_0 - \tfrac{1}{2} H_0.$$

The general solution in terms of three arbitrary functions is

$$H_0 = 2\chi'(x - u_0 t),$$
$$A = F[x - (u_0 + c_0)t] + c_0^{-1}\chi(x - u_0 t) - \tfrac{1}{2} u_0 c_0 E_1/E_0(u_0 + c_0), \quad (11.4)$$
$$B = G[x - (u_0 - c_0)t] + c_0^{-1}\chi(x - u_0 t) - \tfrac{1}{2} u_0 c_0 E_1/E_0(u_0 - c_0). \quad (11.5)$$

Consequently, there is a perturbation, due to the entropy variations, which travels along the particle paths, and this is measured by χ; the perturbations measured by F travel along one family of characteristics with sonic velocity relative to the fluid, and the perturbations measured by G travel along the other family of characteristics with sonic velocity relative to the fluid.

Chester (1953) has considered the disturbance produced behind a plane shock of arbitrary strength travelling down a two-dimensional channel of non-uniform width, and the problem was linearized on the basis of small variations of the width of the channel. The variations in width were assumed to take place within a finite length of the channel, and this region of transition separates two uniform portions each of infinite length but not necessarily equal widths. The pressure field behind the shock was built up from the known solution of the diffraction of a shock wave travelling along a wall with a corner (Lighthill 1948). In a later paper, Chester (1954) considered essentially the same problem, but a different approach was used and applied to a tube of arbitrary cross-section.

Let the flow in front of the shock be given by $u_0 = 0$, c_0, P_0, ρ_0, the flow behind by u_2, c_2, P_2, ρ_2, and the shock velocity by w. Let $v_2 = u_2 - w$ and $M_0 = w/c_0$. The following relations are easily obtained:

$$u_2 = 2w(1-M_0^{-2})/(\gamma+1),$$
$$|v_2| = |u_2 - w| = w(\gamma - 1 + 2M_0^{-2})/(\gamma+1),$$
$$(\gamma+1)P_2 = 2\rho_0[w^2 + (1-\gamma)c_0^2/2\gamma] = P_0[2\gamma M_0^2 - \gamma + 1],$$
$$M_1 = |v_2|/c_2 = [(\gamma - 1 + 2M_0^{-2})/(2\gamma - (\gamma-1)M_0^{-2})]^{1/2},$$
$$m = u_2/c_2 = 2(1-M_1^2)/M_1(\gamma+1).$$

Chester restricted his treatment to the investigation of the average pressure and, for the case where the tube consists of two cylinders connected by a transition section, the following result was obtained:

$$\bar{P}_2 = -K^*(P_2 - P_0)E_1/E_0,$$
$$K^* = 2(1+m)^{-1}(1+M_0^{-2}+2M_1)^{-1}.$$

The parameter K^* decreases monotonically with the shock strength, and $0{\cdot}5 \geqslant K^* > 0{\cdot}394$. Actually, it is clear that the disturbance depends only on the variations in the area of the tube and not on the actual shape of the cross-section, as will be seen shortly.

It will now be shown that, using the purely one-dimensional analysis of this section, Chester's result can be obtained quite simply. Consider the general solution (11.4) and (11.5). The term in E_1 is due directly to the changes in cross-section, and this disturbance is reflected at the shock and gives rise to the term involving G. From the way Chester has formulated the problem (i.e. with the shock coming from infinity, so to speak), the contribution given by F is not included, for there is no mechanism (e.g. a piston curve) for reflection upstream of the shock which could give rise to such a term. In other words, Chester's result should be obtainable from (11.4) and (11.5) by putting $F = 0$. The details are sketched below. If $F = 0$, it follows from (11.4) that

$$\frac{\bar{u}_2}{2c_2} + \frac{\bar{c}_2}{c_2(\gamma-1)} - \frac{\bar{s}_2}{2c_v\gamma(\gamma-1)} = -\frac{m(1+m)^{-1}E_1}{2E_0}.$$

From the shock perturbation tables given in Appendix III, it is easily determined that

$$\bar{c}_2/c_2(\gamma-1) - \tfrac{1}{2}\bar{s}_2/c_v\gamma(\gamma-1) = \tfrac{1}{2}\bar{P}_2/\gamma P_2.$$

Hence

$$\bar{u}_2/c_2 = -\bar{P}_2/\gamma P_2 - mE_1/(1+m)E_0. \qquad (11.6)$$

As

$$u_2/c_0 = 2(M_0 - M_0^{-1})/(\gamma+1), \qquad (\gamma+1)\bar{u}_2/c_0 = 2(1+M_0^{-2})\bar{M}_0,$$

and, from relation IX of Appendix III,

$$\bar{P}_2/P_2 = 4\gamma\bar{M}_0 M_0/(2\gamma M_0^2 - \gamma + 1),$$

then

$$\frac{\bar{u}_2}{c_0} = \frac{(1+M_0^2)(2\gamma M_0^2 - \gamma + 1)\bar{P}_2}{2\gamma(\gamma+1)M_0^3 P_2},$$

$$\frac{c_2^2}{c_0^2} = \frac{P_2\rho_0}{P_0\rho_2} = \frac{(2\gamma M_0^2 - \gamma+1)[2+(\gamma-1)M_0^2]}{M_0^2(\gamma+1)^2}.$$

Hence

$$c_0\bar{u}_2/c_2 c_0 = \bar{u}_2/c_2 = (1+M_0^2)\bar{P}_2/2\gamma M_0^2 M_1 P_2.$$

Also, from (11.6),

$$\bar{P}_2(1+M_0^2+2M_0^2M_1)/2P_2M_0^2 = -\gamma M_1 m E_1/(1+m)E_0.$$

Finally, since

$$P_2 = \{2\gamma M_0^2 - (\gamma-1)\}(P_2-P_0)/2\gamma(M_0^2-1),$$

then

$$\bar{P}_2 = -(P_2-P_0)E_1E_0^{-1}(1+m)^{-1}(1+2\mathrm{M}_1+M_0^{-2})^{-1}\left\{\frac{mM_1(2\gamma M_0^2-\gamma+1)}{M_0^2-1}\right\}.$$

It is easily shown that the term in braces is equal to 2. Hence, the result is the same as that obtained by Chester.

The analysis of the present paper is useful because it can be applied to a variety of problems. In the next section, a problem where the perturbations are reflected on a piston curve is treated. The analysis could be applied to a problem where the tube is open at one end. In such a problem, reflection at a surface of constant pressure would be encountered.

12. Slowly converging or diverging cross-section

For this problem, $E_1(x) = Kx$, where $K > 0$ for a diverging cross-section and $K < 0$ for a converging cross-section. By a uniform compressive motion, a uniform shock is introduced at $x = 0$. It is assumed that the

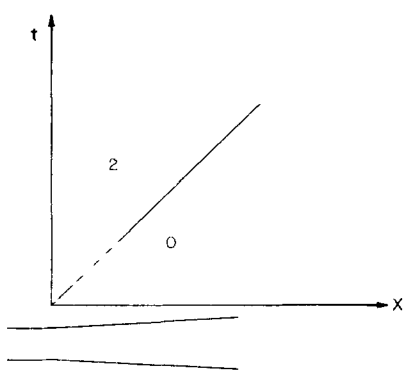

Figure 7.

piston is pushed continuously with velocity V, and the original shock velocity is w. Figure 7 is the (x, t)-representation of the flow. Region O is a rest state, and the perturbed flow behind the shock is denoted by 2.

The problem will be solved by expressing the perturbations $\bar{u}_2$, $\bar{c}_2$ and $\bar{s}_2$ in terms of $\bar{w}$, at present unknown, and using the general solution (§ 11) to find A and B throughout region 2 in terms of $\bar{w}$. From the assumptions of the problem, $\bar{u}_2 = 0$ on the piston path; this gives a functional equation from which $\bar{w}$ can be easily determined.

The Prandtl relation, in perturbation form, gives

$$\bar{u}_2 = 2K_1\bar{w}, \qquad \bar{c}_2 = K_2(\gamma-1)\bar{w}$$

on the shock, with

$$2K_1 = 2(1-\theta) - u_2/w = 2(\gamma+1)^{-1}(1+M_0^{-2}),$$

$$K_2 = \frac{2(w-u_2)}{c_2(\gamma+1)} + \frac{u_2^2}{2wc_2} = \frac{2}{\gamma+1}M_1 + \frac{m}{\gamma+1}(1-M_0^{-2}).$$

Consequently

$$A = (K_1+K_2)\bar{w} \equiv K_3\bar{w}, \qquad B = (K_2-K_1)\bar{w} \equiv K_4\bar{w}$$

on the shock.

By the same analysis as in part II,

$$\bar{s}_2 = \left\{\frac{2c_v K_2(\gamma-1)}{c_2} - \frac{(\gamma-1)c_v[u_2-(1-\theta)w]}{w(u_2-w)}\right\}\bar{w}(t)$$

$$= \frac{c_v(\gamma-1)}{w}\left\{\frac{4}{\gamma+1}\frac{(1-M_1^2)}{(1-M_0^{-2})} + m^2 - 1 + \frac{\gamma-1}{\gamma-1+2M_0^{-2}}\right\} \equiv K_5\bar{w}(t).$$

on the shock. Throughout region 2,

$$\bar{s}_2 = 2c_v\gamma(\gamma-1)c_2^{-2}\chi(x-u_2t),$$

so that

$$2c_v\gamma(\gamma-1)\chi(x-u_2t) = K_5c_2^2\bar{w}[(x-u_2t)/(w-u_2)].$$

By the use of the general solution and the boundary conditions, the solution for region 2 is

$$A = -\frac{u_2c_2Kx}{2E_0(u_2+c_2)} + c_2K_5[2c_v\gamma(\gamma-1)]^{-1}\bar{w}\left[\frac{x-u_2t}{w-u_2}\right] + \left\{K_3 - \frac{c_2K_5}{2c_v\gamma(\gamma-1)}\right\}\bar{w}\left[\frac{x-(u_2+c_2)t}{w-u_2-c_2}\right] + u_2c_2Kw\frac{[x-(u_2+c_2)t]}{[2E_0(u_2+c_2)][w-u_2-c_2]},$$

$$B = -\frac{u_2c_2Kx}{2E_0(u_2-c_2)} + \frac{c_2K_5}{2c_v\gamma(\gamma-1)}\bar{w}\left[\frac{x-u_2t}{w-u_2}\right] + \left[K_4 - \frac{c_2K_5}{2c_v\gamma(\gamma-1)}\right]\bar{w}\left[\frac{x-(u_2-c_2)t}{w-u_2+c_2}\right] + \left[\frac{x-(u_2-c_2)t}{w-u_2+c_2}\right]\frac{u_2c_2Kw}{2E_0(u_2-c_2)}.$$

As $u_2 = 0$ on $x = Vt$, the following equation is obtained for the determination of $\bar{w}$:

$$\left[K_3 - \frac{c_2K_5}{2c_v\gamma(\gamma-1)}\right]\bar{w}\left[\frac{-c_2t}{w-u_2-c_2}\right] + \bar{w}\left[\frac{c_2t}{w-u_2+c_2}\right]\left[\frac{c_2K_5}{2c_v\gamma(\gamma-1)} - K_4\right] = u_2c_2^2K(w-u_2)[(w-u_2)^2-c_2^2]^{-1}E_0^{-1}t.$$

It is easily seen that a linear function of t satisfies this equation, and the solution is

$$\left[\frac{K_5 M_1 c_2}{2c_v \gamma(\gamma-1)} - K_1 - K_2 M_1\right] 2E_0 \bar{w}(t) = u_2 K(w-u_2)t.$$

The pressure perturbation behind the shock is, in terms of the parameters M_0, M_1 and m,

$$\bar{P}_2 = -K_6(P_2 - P_0)E_1 E_0^{-1},$$

$$(\gamma+1)K_6 = 2(\gamma-1+2M_0^{-2})\Bigg[2M_1^2 + mM_1(1-M_0^{-2}) + (1+M_0^{-2}) - \frac{(\gamma-1+2M_0^{-2})}{2\gamma}\left\{\frac{4(1-M_1^2)}{(\gamma+1)(1-M_0^{-2})} + m^2 - 1 + \frac{\gamma-1}{(\gamma-1+2M^{-2})}\right\}\Bigg]^{-1}.$$

Here K_6 is a monotonically decreasing function of the shock strength; and, for $\gamma = 7/5$, $0{\cdot}259 < K_6 \leqslant 0{\cdot}608$. A graph of this function is given in figure 8.

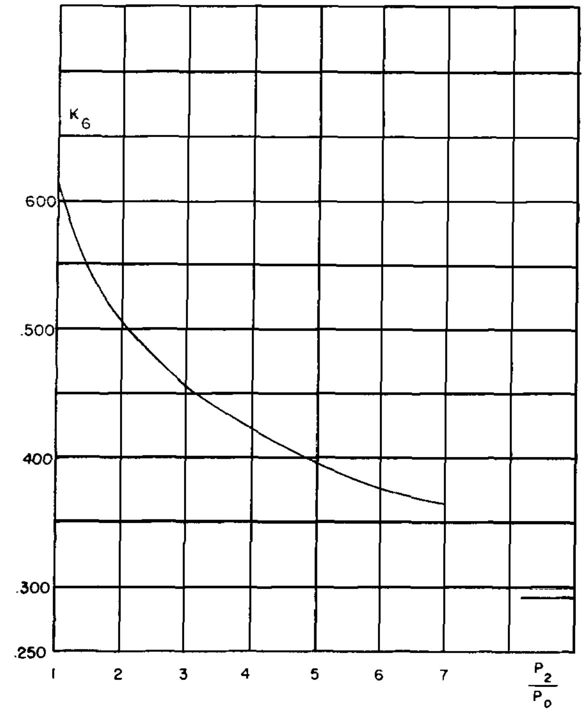

Figure 8.

13. Conclusions

In this work, several problems of flows with weak entropy changes have been considered and solved explicitly. Without going further into the details of the problems treated, we may note the type of functional equations to which the solutions of these problems lead. That is, the values of the function to be found appeared in the same relation for different values of the independent variable.

It is clear that in cases where an analytic solution cannot be found, the general equations are suitable for a numerical calculation developed step by step along the characteristics. This procedure would be necessary for the perturbation of a flow with a shock of constant intensity.

It is also clear that the problems treated here are among the most simple non-isentropic flows that can be considered. Undoubtedly, an analogous theory could be developed for steady flow in two dimensions. However, it is not at all sure that analytic results as complete as the present ones could be obtained. Perturbations can also be considered which introduce a new coordinate, but the problem is then even more complex. Several particular cases where the perturbed flow is uniform have already been considered, but much important work remains to be done.

The following problem, which has already been solved (Gundersen 1954), may be noted. By the use of the method of characteristic coordinates, an analytic solution was obtained for the interaction, in an inviscid ideal gas with a constant ratio of specific heats of 5/3, of a centred rarefaction wave and a non-uniform shock of constant intensity, i.e. the entropy jump is constant all along the shock with the result that the flow is isentropic on both sides of the shock. This work shows that the analysis of this paper need not be restricted to the perturbation of uniform shocks.

These studies are important because flows with shock waves play an increasingly important role in numerous technical problems, and, except for certain exceptional cases, such flows are not isentropic.

The problem discussed in this paper was suggested by Professor Paul Germain and the research directed by him. The author wishes to express his sincere appreciation to Professor Germain for his valuable advice and his continuous and stimulating interest in the project.

Appendix I

The functional equation

$$Af(at)+Bf(bt)=g(t) \tag{A 1}$$

will be considered, where A, B, a and b are constants. Assume

$$|B|>|A|, \qquad |b|>|a|, \tag{A 2}$$

and put

$$\left.\begin{aligned} a/b=\xi, \quad A/B=K, \quad \tau=bt, \quad f(bt)=F(\tau), \quad g(\tau/b)=BG(\tau), \\ f(at)=f(bta/b)=F(\xi\tau). \end{aligned}\right\} \tag{A 3}$$

Consequently, (A 1) may be written as

$$K\,F(\xi\tau)+F(\tau)=G(\tau), \qquad |\xi|<1, \quad |K|<1. \tag{A 4}$$

Conversely, if $G(\tau)$ is known, and if $F(\tau)$ satisfies (A 4), i.e. if (A 4) can be solved, (A 1) can also be solved.

Theorem: If $G(\tau)$ is continuous in an interval I $(-N\leqslant\tau\leqslant N)$, which implies that $G(\tau)$ is also bounded in I, and if $F(\tau)$ is uniformly bounded in a neighbourhood of $\tau = 0$, there exists one and only one solution of (A 4) defined in I. $F(\tau)$ is a continuous function in I. If $G(\tau)$ is continuously differentiable in I, $F(\tau)$ is also continuously differentiable in I.

Proof: If $\tau \in I$, $\xi^n \tau \in I$ because $|\xi| < 1$. Then

$$\begin{aligned} F(\tau) + K\,F(\xi\tau) &= G(\tau), \\ F(\xi\tau) + K\,F(\xi^2\tau) &= G(\xi\tau), \\ \cdots\cdots\cdots\cdots &, \\ F(\xi^n\tau) + K\,F(\xi^{n+1}\tau) &= G(\xi^n\tau). \end{aligned}$$

Consequently,

$$F(\tau) + (-1)^n K^{n+1} F(\xi^{n+1}\tau) = G(\tau) + \sum_{p=1}^{n} (-1)^p K^p G(\xi^p \tau) = g_n(\tau).$$

Hence

$$|F(\tau) - g_n(\tau)| = |K^{n+1}||F(\xi^{n+1}\tau)|.$$

As $F(\tau)$ is to be uniformly bounded in a neighbourhood of $\tau = 0$, it is possible to find n large enough so that $|F(\xi^{n+1}\tau)|$ is bounded. Call this bound C. Thus

$$|F(\tau) - g_n(\tau)| \leqslant C|K^{n+1}|. \tag{A 5}$$

Since $|K| < 1$, if such a solution $F(\tau)$ exists, it must be the uniform limit of the sequence $g_n(\tau)$ in I (uniform convergence). Conversely, if $g_n(\tau)$ is a uniform sequence of functions in I, the limit is the unique solution of (A 4). However, it is obvious that the series

$$\sum_{p=0}^{\infty} (-1)^p K^p\, G(\xi^p \tau)$$

is uniformly convergent in I. Hence the solution of (A 4) is

$$F(\tau) = \sum_{p=0}^{\infty} (-1)^p A^p B^{-p-1} g(a^p \tau / b^{p+1}).$$

Remarks: 1. If $G(\tau) = \tau^n$, $F(\tau) = \tau^n/(1 + K\xi^n)$, $n \geqslant 0$. Thus, if $G(\tau)$ may be approximated by $\sum_{p=1}^{n} d_p\, \tau^p$, $F(\tau)$ may be approximated by

$$\sum_{p=1}^{n} d_p\, \tau^p/(1 + K\xi^p).$$

If $G(\tau)$ has a Taylor expansion in I (i.e. it is an analytic function), $F(\tau)$ is an analytic function in I.

2. If $|K| = 1$, it is necessary to assume $G(0) = 0$. To preserve uniqueness, it is necessary to assume $f(t)$ is continuous near $t = 0$.

3. If $|\xi| = 1$, there is no problem if $\xi = 1$, $K \neq -1$;

$$F(\tau) = G(\tau)/(1 + K).$$

4. If $\xi = -1$, $|K| < 1$, then

$$K\,F(-\tau) + F(\tau) = G(\tau), \qquad K\,F(\tau) + F(-\tau) = G(-\tau),$$
$$F(\tau)(1 - K^2) = G(\tau) + K\,G(-\tau).$$

5. If $\xi = 1$, $|K| = 1$, the problem is either impossible or indeterminate.

Appendix II

The functional equation

$$f(t)+hf(mt)+Kf(nt) = g(t) \tag{B 1}$$

will be considered, where h, K, m and n are constants, and $|m| < 1$, $|n| < 1$, $|h|+|K| < 1$.

Theorem: If $f(t)$ is uniformly bounded in a neighbourhood of $t = 0$, and $g(t)$ is continuous in an interval I ($|t| \leqslant A$), there exists one and only one solution of (B 1) defined in I. $f(t)$ is a continuous function in I and, if $g(t)$ is continuously differentiable in I, $f(t)$ is also continuously differentiable in I. The solution of (B 1) is

$$f(t) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q}h^pK^q g(m^pn^qt)(p+q)!/p!\,q!, \tag{B 2}$$

where this series is uniformly convergent.

First, formally consider the functions $g(m^pn^qt)$, where $p, q = 0, 1, 2, 3, \ldots$, and attempt a solution of (B 1) of the form

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} e_{p,q}\, g(m^pn^qt). \tag{B 3}$$

The series (B 3) is a solution of (B 1) if

$$\left.\begin{aligned} e_{0,0} = 1, \qquad e_{0,1}+Ke_{0,0} = 0, \qquad e_{1,0}+he_{0,0} = 0, \\ e_{p,q}+he_{p-1,q}+Ke_{p,q-1} = 0. \end{aligned}\right\} \tag{B 4}$$

Put

$$e_{p,q} = (-1)^{p+q}d_{p,q}h^pK^q.$$

Then

$$d_{p,q} = d_{p-1,q}+d_{p,q-1}, \qquad d_{0,0} = 1, \qquad d_{0,1} = d_{1,0} = 1,$$

and therefore

$$d_{p,q} = (p+q)!/p!\,q!. \tag{B 5}$$

It is now necessary to show that (B 2) is actually the solution of (B 1) and that it is unique. Consider the operator L_r which is defined by

$$L_r\phi = \sum_{p=0}\sum_{q=0}(-1)^{p+q}h^pK^q\phi(m^pn^qt)(p+q)!/p!\,q!, \qquad p+q \leqslant r. \tag{B 6}$$

The application of this operator to (B 1) gives

$$f(t)-L_rg(t) = \sum_{p+q=r+1}\sum h^pK^q[hf(m^{p+1}n^qt)+Kf(m^pn^{q+1}t)](-1)^{p+q}\frac{(p+q)!}{p!\,q!},$$

$$|f-L_rg| = \sum_{p+q=r+1}\sum [|h||f(m^{p+1}n^qt)|+|K||f(m^pn^{q+1}t)|]|h|^p|K|^q\frac{(p+q)!}{p!\,q!}.$$

Since f is to be uniformly bounded in a neighbourhood of $t = 0$, it is possible to find r large enough so that

$$|f(m^pn^{q+1}t)| < M, \qquad |f(m^{p+1}n^qt)| < M,$$

where M is a constant. Therefore

$$|f-L_rg| < M\sum_{p+q=r+1}\sum |h|^p|K|^q(p+q)!/p!\,q!. \tag{B 7}$$

As $r \to \infty$, the right-hand side approaches zero and

$$f(t) = L_\infty g(t) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} h^p K^q g(m^p n^q t)(p+q)!/p!\,q!\,. \quad \text{(B 8)}$$

As g is uniformly bounded in I, say $|g(t)| < N$, $|t| \leqslant A$, the series

$$N[1+|h|+|K|]^{-1} = N \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{p+q} |h|^p |K|^q (p+q)!/p!\,q! \quad \text{(B 9)}$$

is a dominant series for (B 8), and (B 8) is uniformly convergent. It has been tacitly assumed that $|h|+|K| < 1$. For other values of h and K, f must be analytically continued. This question will not be discussed here.

If $g(t) = t^r, f(t) = t^r(1+hm^r+Kn^r)^{-1}$. Thus, if $g(t)$ may be approximated by $\sum_{r=1}^{p} a_r t^r$, $f(t)$ may be approximated by

$$\sum_{r=1}^{p} b_r t^r [1+hm^r+Kn^r]^{-1}.$$

If $g(t)$ is an analytic function in I, $f(t)$ is an analytic function in I.

Appendix III. Table of perturbations

From the equations which govern the flow, many relations between the perturbed quantities may be obtained. A few of these are tabulated below, where all perturbations will be denoted by a bar, e.g. $\bar{c}_1$ where the basic quantity is c_1.

$$c_1^2 = \gamma \exp[(s_1 - s^*)/c_v]\rho_1^{\gamma-1}, \qquad \bar{\rho}_1 = 2\rho_1 \bar{c}_1/c_1(\gamma-1) - \rho_1 \bar{s}_1/c_v(\gamma-1). \quad \text{(I)}$$

$$c_1^2 = \gamma P_1/\rho_1, \qquad \gamma \bar{P}_1 = c_1^2 \bar{\rho}_1 + 2c_1 \rho_1 \bar{c}_1. \quad \text{(II)}$$

$$\frac{P_2}{P_1} = \frac{\rho_2 - \theta\rho_1}{\rho_1 - \theta\rho_2}, \qquad \frac{\bar{P}_2}{P_2} - \frac{\bar{P}_1}{P_1} = \frac{\bar{\rho}_2 - \theta\bar{\rho}_1}{\rho_2 - \theta\rho_1} - \frac{\bar{\rho}_1 - \theta\bar{\rho}_2}{\rho_1 - \theta\rho_2}. \quad \text{(III)}$$

$$P_2/P_1 = (1+\theta)|u_1 - W|^2/c_1^2 - \theta. \quad \text{(III}')$$

$$\frac{\bar{P}_2 + \theta\bar{P}_1}{P_2 + \theta P_1} - \frac{\bar{P}_1}{P_1} = 2\left[\frac{\bar{u}_1 - \bar{W}}{u_1 - W} - \frac{\bar{c}_1}{c_1}\right],$$

$$\text{or } \frac{\bar{P}_2}{P_2} - \frac{\bar{P}_1}{P_1} = \frac{2(P_2 + \theta P_1)}{P_2}\left[\frac{\bar{u}_1 - \bar{W}}{u_1 - W} - \frac{\bar{c}_1}{c_1}\right]. \quad \text{(IV)}$$

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2}, \qquad \frac{\bar{\rho}_2}{\rho_2} + \frac{\bar{u}_2 - \bar{W}}{u_2 - W} = \frac{\bar{\rho}_1}{\rho_1} + \frac{\bar{u}_1 - \bar{W}}{u_1 - W}. \quad \text{(V)}$$

$$(1-\theta)(W-u_1)^2 - (u_2 - u_1)(W - u_1) = (1-\theta)c_1^2.$$

$$2(1-\theta)(W-u_1)(\bar{W} - \bar{u}_1) - (u_2 - u_1)(w - u_1) \times$$
$$\times \left[\frac{\bar{u}_2 - \bar{u}_1}{u_2 - u_1} + \frac{\bar{W} - \bar{u}_1}{W - u_1}\right] = 2(1-\theta)c_1 \bar{c}_1. \quad \text{(VI)}$$

$$P = \exp[(s - s^*)/c_v]\rho^\gamma, \qquad c^2 = \gamma P/\rho = \gamma P^{(\gamma-1)/\gamma} \exp[(s-s^*)/c_v].$$

$$2\bar{c}_1/c_1 = (1-\gamma^{-1})\bar{P}_1/P_1 + \bar{s}_1/\gamma c_v. \quad \text{(VII)}$$

$$v_2 v_1 = c_*^2 = \theta v_1^2 + (1-\theta)c_1^2 = \theta v_2^2 + (1-\theta)c_2^2.$$

$$(u_1 - W)(\bar{u}_2 - \bar{W}) + (u_2 - W)(\bar{u}_1 - \bar{W}) = 2\theta(u_1 - W)(\bar{u}_1 - \bar{W}) + (1-\theta)2c_1 \bar{c}_1$$
$$= 2\theta(u_2 - W)(\bar{u}_2 - \bar{W}) + (1-\theta)c_2 \bar{c}_2^2. \quad \text{(VIII)}$$

In some problems, it is convenient to express all perturbed quantities in terms of one parameter, e.g. as in the problem of § 4 where $\bar{W}$ was used as the parameter. Another convenient parameter, which leads to a concise representation of the perturbed quantities, is $M = |v|/c$. From relation III′ above,

$$\frac{\bar{P}_2}{P_2} - \frac{\bar{P}_1}{P_1} = \frac{4\gamma M_1 \bar{M}_1}{2\gamma M_1^2 - (\gamma - 1)}, \tag{IX}$$

Now

$$\frac{\rho_2}{\rho_1} = \frac{P_2/P_1 + \theta}{1 + \theta P_2/P_1} = \frac{2\gamma M_1^2/(\gamma+1)}{1 + \theta\{2\gamma M_1/(\gamma+1) - \theta\}}$$

implies that

$$\frac{\bar{\rho}_2}{\rho_2} - \frac{\bar{\rho}_1}{\rho_1} = \frac{4\bar{M}_1}{M_1[2 + (\gamma-1)M_1^2]} = -\frac{\bar{v}_2}{v_2} + \frac{\bar{v}_1}{v_1} \tag{X}$$

in view of the relation $\rho_2 v_2 = \rho_1 v_1$. From $c^2 = \gamma P/\rho$,

$$2\bar{c}_2/c_2 - 2\bar{c}_1/c_1 = \bar{P}_2/P_2 - \bar{P}_1/P_1 - [\bar{\rho}_2/\rho_2 - \bar{\rho}_1/\rho_1],$$

or

$$\frac{\bar{c}_2}{c_2} - \frac{\bar{c}_1}{c_2} = 2(\gamma - 1)\frac{\bar{M}_1}{M_1}\frac{(1 + \gamma M_1^4)}{(2\gamma M_1^2 - \gamma + 1)[2 + (\gamma-1)M]_1^2}. \tag{XI}$$

From (III′),

$$2\gamma M_1/(\gamma+1) - \theta = [-\theta + 2\gamma M_2^2/(\gamma+1)]^{-1},$$

or

$$\bar{M}_1 M_1(\gamma - 1 - 2\gamma M_2^2) = \bar{M}_2 M_2(2\gamma M_1^2 - \gamma + 1). \tag{XII}$$

From the equation of state,

$$\exp[(s_2 - s_1)/c_v] = (P_2/P_1)(\rho_1/\rho_2)^\gamma,$$

or

$$\frac{\bar{s}_2 - \bar{s}_1}{c_v} = \frac{\bar{P}_2}{P_2} - \frac{\bar{P}_1}{P_1} - \gamma\left[\frac{\bar{\rho}_2}{\rho_2} - \frac{\bar{\rho}_1}{\rho_1}\right],$$

it follows that

$$\bar{s}_2 - \bar{s}_1 = \frac{4\gamma c_v(\gamma - 1)(1 - M_1^2)\bar{M}_1}{M_1[2\gamma M_1^2 - \gamma + 1][2 + (\gamma-1)M_1^2]}. \tag{XIII}$$

From $M_2 = v_2/c_2$,

$$\bar{M}_2 = (v_2/c_2)[(\bar{v}_2/v_2) - (\bar{c}_2/c_2)], \tag{XIV}$$

$$\bar{v}_2 = \bar{u}_2 - \bar{W}, \qquad \bar{v}_1 = \bar{u}_1 - \bar{W}. \tag{XV}$$

For a particular shock problem, the basic flow (i.e. all terms without bars) and the perturbed flow (subscript 1) are assumed known. If the problem is to be solved in terms of $\bar{M}_1$, equations (IX) to (XIII) yield $\bar{P}_2$, $\bar{\rho}_2$, $\bar{c}_2$, $\bar{M}_2$, $\bar{s}_2$. From (XIV), $\bar{v}_2$ is obtained, and $\bar{u}_2$ follows from (XV).

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